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On conformal invariant integrals involving spin one-half and spin-one particles

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Abstract
We consider the evaluation of D-dimensional conformal invariant integrals which involve spin one-half and spin-one particles. The star–triangle relation for the massless Yukawa theory is derived, and the longitudinal part of the three-point Green function of massless QED is determined to the lowest order in position space. The operator algebraic method of calculating massless Feynman integrals is used for the evaluation.

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1. Introduction

This work is concerned with the evaluation of conformal invariant integrals in Euclidean space with a general number of dimensions. A scale and Poincaré invariant field theory is also generally conformal invariant. Now, any theory of massless particles with dimensionless couplings is scale invariant at the tree level. Therefore, tree-level integrals in position space in such a theory will exhibit a conformal invariant structure. The simplest example of this is the star–triangle relation (also called the uniqueness relation) involving three massless scalar fields. This relation, which evaluates the integral for the tree-level three-point function, not only brings out the conformal structure, but also evaluates the coefficient exactly. The star–triangle relation in three dimensions was given in [1], and was proved for a general number of dimensions by Symanzik in [2].

Now the conformal invariance of the tree level can get broken due to diverging loop contributions. Even then, such exact relations at the tree level are useful for carrying out the integration over the internal vertices in position space diagrams. Of particular importance, however, is the application of such relations to conformal field theories (CFTs). Various aspects of CFTs in D dimensions have been reviewed in [3, 4]. The evaluation of the tree level integrals is necessary for implementing the bootstrap program in a CFT [5, 6]. Using the star–triangle relation to integrate over the internal vertices, the bootstrap program has been carried out to determine the anomalous dimensions in ϕ^4 theory [7]. However, D-dimensional CFTs generally involve particles with non-zero spin. A well-known example is the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. Calculation of Feynman integrals in the position space in this theory has been carried out in several recent works [8]. In the context of massless QED, a formulation of conformal QED_4 was suggested in [9, 10]. Also, the infrared limit of massless QED_3 is a CFT [11, 12]. Possible application in these theories is a motivation for studying conformal invariant integrals with spinors and vector particles.

Other than this, analytical evaluation of massless Feynman integrals at multi-loop level and the star–triangle relation are generally important for calculations in perturbative field theory at high orders, and in mathematical physics: see [13] and references therein. Recently, a simple method of doing these calculations has been developed by Isaev [13, 14] which replaces the Feynman integrals by algebraic manipulation of operators. We use this method extensively in this paper.

Three-point functions involving conserved vector operators in D-dimensional CFTs have been discussed in [15]. We will, however, be concerned with the three-point function involving the fermion and the gauge field in QED. This has been discussed in [9, 10], and we will compare our result with that given in these two works.

This paper is organized as follows. In section 2, we discuss the usual star–triangle relation using the operator approach. In section 3, we derive the star–triangle relation for the massless Yukawa theory. In section 4, we perform an explicit calculation of the longitudinal part of the three-point Green function of massless QED to the lowest order in position space. Our conclusions are presented in section 5.

2. Star–triangle relation involving scalar fields

It will be helpful to first discuss, following [13], the usual star–triangle relation involving scalar fields within the framework of the operator algebraic method. The relation evaluates $\langle 0|T(\phi_1(x_1)\phi_2(x_2)\phi_3(x_3))|0\rangle$ to the lowest order in position space with a $\phi_1\phi_2\phi_3$ interaction. With $x_{ab} \equiv x_a - x_b$, the relation is given by [2]

$$\int \mathrm{d}^D x_4 \left(x_{14}^2\right)^{-\delta_1} \left(x_{24}^2\right)^{-\delta_2} \left(x_{34}^2\right)^{-\delta_3} = \pi^{D/2} \frac{\Gamma(D/2-\delta_1)\Gamma(D/2-\delta_2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\delta_3)} \times \left(x_{12}^2\right)^{-D/2+\delta_3} \left(x_{13}^2\right)^{-D/2+\delta_2} \left(x_{23}^2\right)^{-D/2+\delta_1}, \tag{1}$$

where

$$\delta_1 + \delta_2 + \delta_3 = D. \tag{2}$$

The left-hand side of equation (1) represents the propagation of a massless scalar particle between the point x_a and the internal vertex x_4 with a scale dimension δ_a, for $a = 1, 2, 3$. It is to be noted that equation (2) ensures that the coupling constant of the $\phi_1\phi_2\phi_3$ interaction is dimensionless, and that the right-hand side of equation (1) has the conformal structure of the three-point function involving three scalar fields also because of equation (2).

In the operator approach, one reduces Feynman integrals to products of position and momentum operators $\hat{q}_i$ and $\hat{p}_i$ ($i = 1, \ldots, D$) taken between position eigenstates. A collection of useful formulae is given in the appendix of our paper. The *key relation* (equation (9) of [13]) is

$$\hat{p}^{-2\alpha}\hat{q}^{-2(\alpha+\beta)}\hat{p}^{-2\beta} = \hat{q}^{-2\beta}\hat{p}^{-2(\alpha+\beta)}\hat{q}^{-2\alpha}. \tag{3}$$

This is *the star–triangle relation in the operator form*. To see this, one has to take equation (3) between the states $\langle x|$ and $|y\rangle$. This gives, on inserting the completeness relation and using equations (A.4), (A.5) and (A.6),

$$\int \mathrm{d}^D z \frac{1}{|x-z|^{D-2\alpha}} \frac{1}{|z|^{2(\alpha+\beta)}} \frac{1}{|y-z|^{D-2\beta}} = \pi^{D/2} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(D/2-\alpha-\beta)}{\Gamma(\alpha+\beta)\Gamma(D/2-\alpha)\Gamma(D/2-\beta)} \times \frac{1}{|x|^{2\beta}} \frac{1}{|x-y|^{D-2\alpha-2\beta}} \frac{1}{|y|^{2\alpha}}. \tag{4}$$

It is important to note from equations (3) and (4) that the '$\hat{p}\hat{q}\hat{p}$' form represents the integral, while the '$\hat{q}\hat{p}\hat{q}$' form gives the result of integration. Now let $x = x_1 - x_2$ and $y = x_3 - x_2$, and let us also change to a new integration variable x_4 defined by $z = x_4 - x_2$. Also, define δ_1, δ_2 and δ_3 by $D/2 - \alpha = \delta_1, \alpha + \beta = \delta_2$ and $D/2 - \beta = \delta_3$. This leads us to the relation stated in the form of equation (1).

3. Star–triangle relation for massless Yukawa theory

We now turn to the massless Yukawa theory with a $\bar{\psi}\psi\phi$ interaction and a dimensionless coupling. Symanzik [2] gives the method of deriving the star–triangle relation for this theory using Schwinger parameters. We show how the operator approach provides us with an alternative method, and derive the relation. The manipulations which we perform also set the stage for the calculation of section 4.

The suitable starting '$\hat{p}\hat{q}\hat{p}$' form is now

$$\Gamma \equiv \gamma_i \gamma_j \hat{p}_i \hat{p}^{-2\alpha-1} \hat{q}_j \hat{q}^{-2(\alpha+\beta)-1} \hat{p}^{-2\beta}, \tag{5}$$

so that

$$\langle x|\Gamma|y\rangle = \mathrm{i}(D-2\alpha-1)a(\alpha+1/2)a(\beta) \int \mathrm{d}^D z \frac{\not{x} - \not{z}}{|x-z|^{D-2\alpha+1}} \frac{\not{z}}{|z|^{2(\alpha+\beta)+1}} \frac{1}{|y-z|^{D-2\beta}}. \tag{6}$$

To convert Γ to '$\hat{q}\hat{p}\hat{q}$' form, we first put $\hat{q}_j$ next to $\hat{p}_i$ in equation (5) by using equation (A.2). (We extend the validity of equation (A.2) to all real α, and use it with $-2\alpha - 1$ in the place of 2α.) We thus obtain

$$\Gamma = \gamma_i \gamma_j \hat{p}_i (\hat{q}_j \hat{p}^{-2\alpha-1} + \mathrm{i}(2\alpha+1)\hat{p}^{-2\alpha-3}\hat{p}_j) \hat{q}^{-2(\alpha+\beta)-1} \hat{p}^{-2\beta}. \tag{7}$$

Since $\gamma_i \gamma_j \hat{p}_i \hat{p}_j = \hat{p}^2$, we can use the key relation of equation (3) (with $2\alpha + 1$ in the place of 2α) in both the terms and obtain

$$\Gamma = \gamma_i \gamma_j \hat{p}_i \hat{q}_j \hat{q}^{-2\beta} \hat{p}^{-2(\alpha+\beta)-1} \hat{q}^{-2\alpha-1} + \mathrm{i}(2\alpha+1)\hat{q}^{-2\beta} \hat{p}^{-2(\alpha+\beta)-1} \hat{q}^{-2\alpha-1}. \tag{8}$$

The second term is already of '$\hat{q}\hat{p}\hat{q}$' form. To put the first term also into that form, $\hat{p}_i$ has to be brought next to $\hat{p}^{-2(\alpha+\beta)-1}$. For this, we take $\hat{p}_i$ first through $\hat{q}_j$ using equation (A.1) and then through $\hat{q}^{-2\beta}$ using equation (A.3). This generates a couple of additional terms, and on simplification equation (8) reduces to

$$\Gamma = \gamma_i \gamma_j \hat{q}_j \hat{q}^{-2\beta} \hat{p}_i \hat{p}^{-2(\alpha+\beta)-1} \hat{q}^{-2\alpha-1} - \mathrm{i}(D-2\alpha-2\beta-1)\hat{q}^{-2\beta} \hat{p}^{-2(\alpha+\beta)-1} \hat{q}^{-2\alpha-1}. \tag{9}$$

To obtain $\langle x|\Gamma|y\rangle$, we use equations (A.4), (A.5) and (A.7). This gives

$$\langle x|\Gamma|y\rangle = \mathrm{i}(D-2\alpha-2\beta-1)a(\alpha+\beta+1/2) \frac{(\not{x} - \not{y})\,\not{y}}{|x|^{2\beta}|x-y|^{D-2\alpha-2\beta+1}|y|^{2\alpha+1}}. \tag{10}$$

Equating the right-hand sides of equations (6) and (10), we arrive at the desired relation. The coefficients can be determined from equation (A.6) and simplified using the relation

$n\Gamma(n) = \Gamma(n+1)$. Finally, using the new variables x_a and δ_a as below equation (4), we arrive at the form

$$\int \mathrm{d}^D x_4 \frac{\not{x}_{14}}{\left(x_{14}^2\right)^{\delta_1+1/2}} \frac{\not{x}_{42}}{\left(x_{24}^2\right)^{\delta_2+1/2}} \frac{1}{\left(x_{34}^2\right)^{\delta_3}}$$
$$= \pi^{D/2} \frac{\Gamma(D/2-\delta_1+1/2)\Gamma(D/2-\delta_2+1/2)\Gamma(D/2-\delta_3)}{\Gamma(\delta_1+1/2)\Gamma(\delta_2+1/2)\Gamma(\delta_3)}$$
$$\times \frac{\not{x}_{13}}{\left(x_{13}^2\right)^{D/2-\delta_2+1/2}} \frac{\not{x}_{32}}{\left(x_{23}^2\right)^{D/2-\delta_1+1/2}} \frac{1}{\left(x_{12}^2\right)^{D/2-\delta_3}}. \tag{11}$$

As before, equation (2) ensures scale invariance at the tree level. For the special case of $D = 4$, equation (11) is in agreement with equation (A6.12a) of [3].

4. Three-point Green function for massless QED

In order to evaluate $\langle 0|T(\psi(x_1)\bar{\psi}(x_2)A_k(x_3)|0\rangle$ at the lowest order, we first need to specify the tree-level propagators in the position space. We follow the convention [3, 4] of writing the behavior of the fermion propagator and the photon propagator as

$$S(x) \sim \frac{\not{x}}{(x^2)^{d_\psi+1/2}}, \qquad D_{kl}(x) \sim \left(\delta_{kl} - (1-\eta)\frac{\partial_k\partial_l}{\partial^2}\right)\frac{1}{(x^2)^{d_A}}. \tag{12}$$

Here d_ψ and d_A are the scale dimensions, and η is the gauge-fixing parameter. The fermion propagator $\not{p}/p^2$ in momentum space implies $d_\psi = (D-1)/2$. For the photon, we will consider $d_A = 1$ [9, 10]. A photon propagator thus behaving as $1/p^{D-2}$ in momentum space ensures that the QED coupling constant is dimensionless. This behavior, present in QED$_4$, also occurs in massless QED$_3$ in the infrared: in the latter theory, the photon propagator goes as $1/p$ in the infrared in the $1/N$ expansion (N being the number of fermion flavors) [11, 16].

The starting '$\hat{p}\hat{q}\hat{p}$' form for the lowest-order three-point function is therefore

$$\hat{p}_i\gamma_i\hat{p}^{-2}\gamma_l\hat{q}_j\gamma_j\hat{q}^{-D}\hat{p}^{-D+2}(\delta_{kl} - (1-\eta)\hat{p}_k\hat{p}_l\hat{p}^{-2}). \tag{13}$$

(It may be helpful to compare equation (13) with equation (5). In equation (13), we have $\alpha = 1/2$ and $\beta = (D-2)/2$. There is also a γ_l vertex factor and the tensor part of the photon propagator.) In the present work, we will consider only the *longitudinal* part:

$$\Gamma_k \equiv \eta\gamma_i\gamma_l\gamma_j\hat{p}_i\hat{p}^{-2}\hat{q}_j\hat{q}^{-D}\hat{p}^{-D+2}\hat{p}_k\hat{p}_l\hat{p}^{-2}. \tag{14}$$

On using the position space 'matrix elements' of $\hat{p}_i\hat{p}^{-2}$, $\hat{p}^{-D+2}$ and $\hat{p}_k\hat{p}_l\hat{p}^{-2}$ from equations (A.7), (A.5) and (A.8) respectively, we find that

$$\langle x|\Gamma_k|y\rangle = \mathrm{i}\eta\frac{(D-2)}{(2\pi)^D}\int \mathrm{d}^D z \frac{\not{x}-\not{z}}{|x-z|^D}\gamma_l\frac{\not{z}}{|z|^D}\frac{\partial_k^y\partial_l^y}{(\partial^2)^y}\frac{1}{|y-z|^2}. \tag{15}$$

Our aim is to simplify equation (14) using the basic identity given in equation (3). The problem in doing this is that it would lead us to (as the calculation given later shows) applying equation (3) on $\hat{p}^{-2}\hat{q}^{-D}\hat{p}^{-D+2}$. This (naively) results in $\hat{q}^{-D+2}\hat{p}^{-D}\hat{q}^{-2}$. But using equations (A.5) and (A.6) for $\langle x|\hat{p}^{-D}|y\rangle$ is not possible because $\Gamma(D/2-\alpha)$ blows up for $\alpha = D/2$.

This problem can be solved by the following regularization of the scale dimensions:

$$d_\psi = \frac{D-1-\epsilon}{2}, \qquad d_A = 1+\epsilon. \tag{16}$$

This corresponds to the propagators $\not{x}/x^{D-\epsilon} \sim \not{p}/p^{2+\epsilon}$ and $1/x^{2+2\epsilon} \sim 1/p^{D-2-2\epsilon}$ for the fermion and the photon respectively. The regularization of the two scale dimensions goes together, since the interaction $\bar{\psi}\gamma_i\psi A_i$ must continue to have the dimension D. Our regularization is similar to that given in equation (2.30) of [4], except that we have changed the sign in front of ϵ in both d_ψ and d_A. This has been done to ensure that we are led to $\langle x|\hat{p}^{-D+\epsilon}|y\rangle$ in the course of our calculation (see below), which is convergent (whereas $\langle x|\hat{p}^{-D-\epsilon}|y\rangle$ would diverge in the infrared).

We therefore have to simplify the regularized form of equation (14):

$$\Gamma_k = \eta\gamma_i\gamma_l\gamma_j\hat{p}_i\hat{p}^{-2-\epsilon}\hat{q}_j\hat{q}^{-D+\epsilon}\hat{p}^{-D+2+2\epsilon}\hat{p}_l\hat{p}_k\hat{p}^{-2}. \tag{17}$$

Using equation (A.8) for the 'matrix element' of $\hat{p}_k\hat{p}^{-2}$, we can write

$$\langle x|\Gamma_k|y\rangle = -\mathrm{i}\eta\frac{\partial_k^y}{(\partial^2)^y}\langle x|\Gamma'|y\rangle, \tag{18}$$

$$\Gamma' = \gamma_i\gamma_l\gamma_j\hat{p}_i\hat{p}^{-2-\epsilon}\hat{q}_j\hat{q}^{-D+\epsilon}\hat{p}^{-D+2+2\epsilon}\hat{p}_l. \tag{19}$$

It is easier to put Γ' in '$\hat{q}\hat{p}\hat{q}$' form than Γ_k. First use equation (A.2) to obtain

$$\Gamma' = \gamma_i\gamma_l\gamma_j\hat{p}_i\hat{p}^{-2-\epsilon}\hat{q}^{-D+\epsilon}(\hat{p}^{-D+2+2\epsilon}\hat{q}_j - \mathrm{i}(D-2-2\epsilon)\hat{p}^{-D+2\epsilon}\hat{p}_j)\hat{p}_l. \tag{20}$$

The identity of equation (3) can now be used in both the terms, giving

$$\Gamma' = \gamma_i\gamma_l\gamma_j\hat{p}_i\hat{q}^{-D+2+2\epsilon}\hat{p}^{-D+\epsilon}\hat{q}^{-2-\epsilon}\hat{q}_j\hat{p}_l - \mathrm{i}(D-2-2\epsilon)\gamma_i\hat{p}_i\hat{q}^{-D+2+2\epsilon}\hat{p}^{-D+\epsilon}\hat{q}^{-2-\epsilon}. \tag{21}$$

We now bring $\hat{p}_l$ next to $\hat{p}^{-D+\epsilon}$ in the first term by moving it through $\hat{q}_j$ and then $\hat{q}^{-2-\epsilon}$ by using equations (A.1) and (A.3) respectively. This leads to

$$\Gamma' = \gamma_i\gamma_l\gamma_j\hat{p}_i\hat{q}^{-D+2+2\epsilon}\hat{p}^{-D+\epsilon}\hat{p}_l\hat{q}^{-2-\epsilon}\hat{q}_j + \mathrm{i}\epsilon\gamma_i\hat{p}_i\hat{q}^{-D+2+2\epsilon}\hat{p}^{-D+\epsilon}\hat{q}^{-2-\epsilon}. \tag{22}$$

Finally $\hat{p}_i$ is brought next to $\hat{p}^{-D+\epsilon}$ in both the terms to arrive at the '$\hat{q}\hat{p}\hat{q}$' form:

$$\begin{aligned}\Gamma' = {}& \gamma_i\hat{q}^{-D+2+2\epsilon}\hat{p}^{-D+2+\epsilon}\hat{q}^{-2-\epsilon}\hat{q}_i + \mathrm{i}(D-2-2\epsilon)\gamma_i\gamma_l\gamma_j\hat{q}^{-D+2\epsilon}\hat{q}_i\hat{p}^{-D+\epsilon}\hat{p}_l\hat{q}^{-2-\epsilon}\hat{q}_j \\ &+ \mathrm{i}\epsilon\gamma_i\hat{q}^{-D+2+2\epsilon}\hat{p}^{-D+\epsilon}\hat{p}_i\hat{q}^{-2-\epsilon} - \epsilon(D-2-2\epsilon)\gamma_i\hat{q}^{-D+2\epsilon}\hat{q}_i\hat{p}^{-D+\epsilon}\hat{q}^{-2-\epsilon}.\end{aligned} \tag{23}$$

The evaluation of $\langle x|\Gamma'|y\rangle$ can now be completed by using equations (A.4)–(A.7). It is found that in the resulting terms, the diverging $\Gamma(\epsilon/2)$ always comes multiplied by ϵ. Since $\epsilon\Gamma(\epsilon/2) = 2\Gamma(1+\epsilon/2)$, taking $\epsilon \to 0$ gives finite results for all the four terms of equation (23) (with the third term giving zero). We then obtain

$$\langle x|\Gamma'|y\rangle = \frac{1}{\pi^{D/2}2^{D-2}\Gamma(D/2-1)}\frac{x^2\,\not{y} - \not{x}(\not{x}-\not{y})\,\not{y} - \not{x}|x-y|^2}{x^D|x-y|^2|y|^2} \tag{24}$$

$$= \frac{1}{\pi^{D/2}2^{D-2}\Gamma(D/2-1)}\frac{\not{x}}{x^D}\left(\frac{1}{|x-y|^2} - \frac{1}{|y|^2}\right). \tag{25}$$

Since $(\partial^2)^y\ln(|x-y|/|y|) = (D-2)(1/|x-y|^2 - 1/|y|^2)$, equations (18) and (25) lead to

$$\langle x|\Gamma_k|y\rangle = -\mathrm{i}\eta\frac{1}{(4\pi)^{D/2}\Gamma(D/2)}\frac{\not{x}}{x^D}\partial_k^y\ln\frac{|x-y|^2}{|y|^2}. \tag{26}$$

The right-hand sides of equations (15) and (26) are now to be equated. In terms of the variables x_1, x_2, x_3 and x_4 defined below equation (4), the resulting relation reads

$$\int \mathrm{d}^D x_4 \frac{\not{x}_{14}}{x_{14}^D}\gamma_l \frac{\not{x}_{42}}{x_{24}^D}\frac{\partial_k^{x_3}\partial_l^{x_3}}{(\partial^2)^{x_3}}\frac{1}{x_{34}^2} = \frac{\pi^{D/2}}{(D-2)\Gamma(D/2)}\frac{\not{x}_{12}}{x_{12}^D}\partial_k^{x_3}\ln\frac{x_{23}^2}{x_{13}^2} \tag{27}$$

$$= \frac{2\pi^{D/2}}{(D-2)\Gamma(D/2)}\frac{\not{x}_{12}}{x_{12}^D}\left(\frac{(x_{13})_k}{x_{13}^2} - \frac{(x_{23})_k}{x_{23}^2}\right). \tag{28}$$

Equations (27) and (28) agree with the longitudinal structure function given from general considerations of conformal invariance in [9, 10] respectively (the fermion scale dimension being $d_\psi = (D-1)/2$ in our case). From equation (28), we note the value of the coefficient for the physically interesting cases: π^2 for $D = 4$ (massless QED_4) and 4π for $D = 3$ (massless QED_3 in the infrared).

5. Conclusion

In this work, we evaluated conformal invariant integrals involving spin one-half and spin-one particles in the context of two D-dimensional field theories with tree-level scale invariance: the massless Yukawa theory and massless QED, both with dimensionless coupling constants. The three-point function of the Yukawa theory and the longitudinal part of the three-point function of QED were explicitly evaluated to the lowest order, and the results were expressed in conformal invariant forms. We made use of the operator algebraic method of calculating massless Feynman integrals. For the QED calculation, regularization of the scale dimensions of the particles was used. While the present work focused on the longitudinal part only, our plan is to evaluate the entire QED three-point function to the lowest order. The result can then be used in higher order studies of massless QED_3 in the infrared and also for implementing the bootstrap program in that theory. More generally, the techniques developed in the present work should be useful for calculations in other massless field theories and D-dimensional CFTs.

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Appendix

In this appendix, we list and develop some important formulae of the operator approach to the evaluation of massless Feynman integrals. We use $i, j, k, \ldots$ for spacetime indices, and $\alpha, \beta, \ldots$ for exponents of $\hat{q}^2$ and $\hat{p}^2$. Thus, $\hat{q}^{2\alpha} = \left(\sum_i \hat{q}_i\hat{q}_i\right)^\alpha$ (and likewise $\hat{p}^{2\alpha}$), the parameter α being in general a complex number [13]. The fundamental commutation relation

$$[\hat{q}_i, \hat{p}_j] = \mathrm{i}\delta_{ij} \tag{A.1}$$

leads to the following two useful relations:

$$[\hat{q}_i, \hat{p}^{2\alpha}] = \mathrm{i}2\alpha\hat{p}^{2\alpha-2}\hat{p}_i, \tag{A.2}$$

$$[\hat{p}_i, \hat{q}^{2\alpha}] = -\mathrm{i}2\alpha\hat{q}^{2\alpha-2}\hat{q}_i. \tag{A.3}$$

(A check on equations (A.2) and (A.3) is that they immediately give us equations (13) and (14) of [13].)

We use the normalization of position and momentum eigenstates followed in [13]. This results in the following two 'matrix elements' [13],

$$\langle x|\hat{q}^{2\alpha}|y\rangle = |x|^{2\alpha}\delta^{(D)}(x-y), \tag{A.4}$$

$$\langle x|\hat{p}^{-2\alpha}|y\rangle = a(\alpha)\frac{1}{|x-y|^{D-2\alpha}}, \tag{A.5}$$

where

$$a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\pi^{D/2}2^{2\alpha}\Gamma(\alpha)}. \tag{A.6}$$

In equation (A.5), $D/2-\alpha \neq 0, -1, -2, \ldots$. Now, $\langle x|\hat{p}_i\hat{p}^{-2\alpha}|y\rangle = -\mathrm{i}\partial_i^x\langle x|\hat{p}^{-2\alpha}|y\rangle$ (this being obtained by inserting the completeness relation in momentum space on the left-hand side). Equation (A.5) then gives us

$$\langle x|\hat{p}_i\hat{p}^{-2\alpha}|y\rangle = \mathrm{i}(D-2\alpha)a(\alpha)\frac{(x-y)_i}{|x-y|^{D-2\alpha+2}}. \tag{A.7}$$

Another useful 'matrix element' which can be similarly obtained is $\langle x|\hat{p}_i|y\rangle = \mathrm{i}\partial_i^y\delta^{(D)}(x-y)$. This relation can be generalized to

$$\langle x|f(\hat{p}_i)|y\rangle = f\left(\mathrm{i}\partial_i^y\right)\delta^{(D)}(x-y), \tag{A.8}$$

where f denotes an arbitrary function. As a check, it may be noted that consistency of equation (A.8) with equations (A.5) and (A.6) leads to the expression for the Green function for the operator $((-\partial^2)^y)^\alpha$.

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